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Cayley–Bacharach theorem of piecewise algebraic curves ☆

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Abstract

The piecewise algebraic curve, determined by a bivariate spline function, is a generalization of the classical algebraic curve. In this paper, by using Bezout's theorem and Nöther-type theorem of piecewise algebraic curves, the Cayley–Bacharach theorem and Hilbert function of C^0 piecewise algebraic curves are presented.

Keywords: Bivariate splines; Piecewise algebraic curves; Cayley–Bacharach theorem; Nöther-type theorem; Bezout's theorem

1. Introduction

Let Ω be a domain in $\mathbb{R}^2$, and Δ be a partition of Ω given by finite irreducible algebraic curves. Denote all of the cells of Δ by δ_i, $i = 1, \dots, T$, where T is the number of the cells of Δ. These line segments which form the boundary of each cell are called the edges. Intersection points of the edges are called the vertices. Denote by $\mathbf{P}_d(\Delta)$ the collection of piecewise polynomials of degree d as follows:

$$\mathbf{P}_d(\Delta) := \{p \mid p_i = p|_{\delta_i} \in \mathbf{P}_d, i = 1, 2, \dots, T\},$$

where $\mathbf{P}_d$ denotes the collection of bivariate polynomials with total degree d. For integer $0 \leqslant \mu < d$, we say that

$$S_d^\mu(\Delta) := \{s \in C^\mu(\Omega) \cap \mathbf{P}_d(\Delta)\}$$

is a bivariate spline space with smoothness μ and degree d. The curve

$$\Gamma\colon \{(x, y) \mid f(x, y) = 0, f(x, y) \in S_d^\mu(\Delta)\}$$

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is called a C^μ *piecewise algebraic curve*. For convenience, it is also called a piecewise algebraic curve f. It is obvious that the piecewise algebraic curve is a generalization of the classical algebraic curve [10,11]. The piecewise algebraic curve is not only very important for the interpolation by the bivariate splines [10,11] but also a useful tool for studying traditional algebraic curve [8,10,11,13,17]. Because of the possibility $\{(x,y)\in\Omega | f|_{\delta_i} = f_i(x,y) = 0\} \cap \overline{\delta_i} = \phi$ (empty), it is more difficult to study the piecewise algebraic curve [4–6,9–17].

It is well known that Bezout's and Nöther's theorems are important and classical results in algebraic geometry [3,8]. The weak form of Bezout's theorem says that two algebraic curves will have infinitely many intersection points provided that they have more intersection points than the product of their degrees. Denote by $BN = BN(m,r;n,t;\Delta)$ the so-called Bezout's number. It means any two piecewise algebraic curves

$$f(x,y)=0, g(x,y)=0, f\in S_m^r(\Delta), g\in S_n^t(\Delta),$$

must have infinitely many intersection points provided that they have more than BN intersection points. Wang et al. have studied the Bezout's number in many ways [4,5,13,14,17]. Nöther-type theorems of piecewise algebraic curves, as the generalizations of Nöther's theorem, on the star partition and cross-cut partition were shown in [4,16].

The Cayley–Bacharach theorem is very important in algebraic geometry [2,3,8]. There is a long and interesting history about the result, started with a famous result by Pappus of Alexandria, proved in the fourth century A.D. As the methods and substance of algebraic geometry have changed over the years, the result has been successively generalized, improved, and reinterpreted, and this development continues today [2,3,7,8]. In this paper, by using Bezout's theorem and Nöther-type theorem of piecewise algebraic curves, the Cayley–Bacharach theorem and Hilbert function of C^0 piecewise algebraic curves are presented.

2. Preliminaries

Definition 2.1. Let $f,g\in \mathbf{P}(\Delta)$. If two polynomials $f_i = f|_{\delta_i}$ and $g_i = g|_{\delta_i}$ have no nonconstant common factors, we say that they have no local common factors. If f and g have no local common factors, or f_i and $g_i (i\in I\subset\{1,2,\dots,N\})$ have a common factor, say r_i, and for each such r_i,

$$\{(x,y)|r_i=0\}\cap\overline{\delta_i}=\phi,$$

we say that the two piecewise algebraic curves f and g have no local common components.

Definition 2.2. If $\Gamma=\{p_1,\dots,p_m\}\subset\Omega$ is any collection of m distinct points, we shall say that Γ imposes l conditions on $S_d^\mu(\Delta)$ if in the vector space of $S_d^\mu(\Delta)$ the subspace of those vanishing at $p_1,\dots,p_m$ has codimension l, or equivalently if in the space of C^μ piecewise algebraic curves of degree d the subspace of those containing Γ has codimension l. Denote the number of conditions imposed by Γ on $S_d^\mu(\Delta)$ by $h_\Gamma(d,\mu)$, and h_Γ is called the Hilbert function of Γ.

Lemma 2.1 (Cui [1]). *Let m,n and r be natural numbers. If two curves $p(x,y)=0$ of degree m and $q(x,y)=0$ of degree n meet exactly in mn distinct points*, *and the curve $f(x,y)=0$ of* *degree r passes through these mn distinct points, then there must exist polynomials* $\alpha(x, y) \in \mathbf{P}_{r-m}$, $\beta(x, y) \in \mathbf{P}_{r-n}$, *such that*

$$f(x, y) = \alpha(x, y)p(x, y) + \beta(x, y)q(x, y).$$

If all the cells of Δ share the interior vertex V of Δ as a common point, then the partition is called a star partition, denoted by Δ_V. If all the edges of Δ are straight lines cross-cut the domain Ω, then the partition is called a cross-cut partition, denoted by Δ_c.

Theorem 2.1 (Shi et al. [5,13]). *If Δ is a 2-triangle-signed triangulation, then*

$$BN(m, 0; n, 0; \Delta) = mnT. \tag{1}$$

Remark 2.1. From Theorem 2.1, there must exist partitions, such as Δ_c having no interior vertices, type-1 triangulation and type-2 triangulation etc., satisfy $BN(m, 0; n, 0; \Delta) = mnT$.

Lemma 2.2. *Let m, n and r be natural numbers. Suppose $BN(m, 0; n, 0; \Delta) = mnT$, and two piecewise algebraic curves G and F, $G \in S^0_m(\Delta), F \in S^0_n(\Delta)$, meet exactly in mnT distinct points. If the curve $H = 0$, $H \in \mathbf{P}_r(\Delta)$, passes through these mnT distinct points, then*

$$H = AG + BF,$$

where $A \in \mathbf{P}_{r-m}(\Delta)$, $B \in \mathbf{P}_{r-n}(\Delta)$.

Proof. According to the assumption conditions, the piecewise algebraic curves G and F meet exactly in mn distinct points on each cell of Δ. By using Lemma 2.1, we get the conclusion. □

By using Theorem 2.1, Lemma 2.2, and Nöther-type theorem of piecewise algebraic curves [16], we obtain the following results.

Lemma 2.3. *Let m, n and r be natural numbers. Suppose that the partition Δ has no interior vertices, and two piecewise algebraic curves G and F, $G \in S^0_m(\Delta), F \in S^0_n(\Delta)$, meet exactly in mnT distinct points. If the piecewise algebraic curve H, $H \in S^0_r(\Delta)$, passes through these mnT distinct points, then $H = AG + BF$, where $A \in S^0_{r-m}(\Delta)$, $B \in S^0_{r-n}(\Delta)$.*

Lemma 2.4. *Let m, n and r be natural numbers. Suppose that $BN(m, 0;\ n, 0;\ \Delta_V) = mnT$, and two piecewise algebraic curves G and F, $G \in S^0_m(\Delta_V), F \in S^0_n(\Delta_V)$, meet exactly in mnT distinct points. If the piecewise algebraic curve H, $H \in S^0_r(\Delta_V)$, passes through these mnT distinct points, then $H = AG + BF$, where $A \in S^0_{r-m}(\Delta_V)$, $B \in S^0_{r-n}(\Delta_V)$.*

Lemma 2.5. *Let m, n and r be natural numbers. Suppose that $BN(m, 0; n, 0; \Delta_c) = mnT$, and two piecewise algebraic curves G and F, $G \in S^0_m(\Delta_c), F \in S^0_n(\Delta_c)$, meet exactly in mnT distinct points. If the piecewise algebraic curve H, $H \in S^0_r(\Delta_c)$, passes through these mnT distinct points, then $H = AG + BF$, $A = U + Q^A$, and $B = V - Q^B$, where $A, Q^A \in \mathbf{P}_{r-m}(\Delta_c), B, Q^B \in \mathbf{P}_{r-n}(\Delta_c)$, and $U \in S^0_{r-m}(\Delta_c), V \in S^0_{r-n}(\Delta_c)$.*

3. Cayley–Bacharach theorem of piecewise algebraic curves

Theorem 3.1 (Cayley–Bacharach Theorem). *Let m, n and r be natural numbers, and $3 \leqslant r \leqslant \min\{m, n\} + 2$. Suppose that $BN(m, 0; n, 0; \Delta) = mnT$, and two piecewise algebraic curves G and F, $G \in S_m^0(\Delta), F \in S_n^0(\Delta)$, meet exactly in mnT distinct points. If the piecewise algebraic curve H, $H \in S_r^0(\Delta)$, passes through $Tmn - \dim(S_{r-3}^0(\Delta))$ points of these mnT points, then it must pass through the $\dim(S_{r-3}^0(\Delta))$ remainder points as well, unless these $\dim(S_{r-3}^0(\Delta))$ remainder points lie on a piecewise algebraic curve p, where $p \in S_{r-3}^0(\Delta)/\{0\}$.*

Proof. Let the piecewise algebraic curves G and F meet exactly in the collection of mnT distinct points $\Gamma = \Gamma^1 \cup \cdots \cup \Gamma^{\mathrm{T}}$, where Γ^i denotes the collection of intersection points of $G_i = G|_{\delta_i} = 0$ and $F_i = F|_{\delta_i} = 0$ on the cell $\delta_i, i = 1, \ldots, T$.

(1) For $r = 3$, the number of remainder points is 1. Without lose of generality, suppose that the curve $H_i = H|_{\delta_i} = 0$ contains all but one point of Γ^i. Using Cayley–Bacharach theorem, the curve $H_i = 0$ contains all of Γ^i. Thus piecewise algebraic curve H passes through the remainder point as well.

(2) For $r > 3$, let $\Gamma' = \{Q_i\}_{i=1}^{\dim(S_{r-3}^0(\Delta))}$ be the collection of the remainder points. Since Γ' does not lie on any piecewise algebraic curve p, $p \in S_{r-3}^0(\Delta)/\{0\}$, Γ' must be a properly posed knot set for $S_{r-3}^0(\Delta)$ [10,11]. Then there must exist a collection of spline functions $\{s_i\}_{i=1}^{\dim(S_{r-3}^0(\Delta))} \subset S_{r-3}^0(\Delta)$ satisfying

$$s_i(Q_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i, j = 1, \ldots, \dim(S_{r-3}^0(\Delta)). \tag{2}$$

It is clear that $s_i H \in S_{m+n-3}^0(\Delta)$ and the piecewise algebraic curve $s_i H$ passes through all but one point of Γ. By the proof of the case of $r = 3$, the piecewise algebraic curve $s_i H$ contains all of Γ. By Lemma 2.2, there exists $A_i \in \mathbf{P}_{n-3}(\Delta), B_i \in \mathbf{P}_{m-3}(\Delta)$, such that

$$s_i H = A_i G + B_i F.$$

Hence for any $Q_i \in \Gamma'$, we can see that $s_i(Q_i)H(Q_i) = 0$. Since $s_i(Q_i) = 1 \neq 0$, we have $H(Q_i) = 0$. Taking $i \in \{1, \ldots, \dim(S_{r-3}^0(\Delta))\}$, we get the conclusion. □

Theorem 3.2. *Let m, n and r be natural numbers. Suppose that $BN(m, 0; n, 0; \Delta) = mnT$, two piecewise algebraic curves H and F, $H \in S_m^0(\Delta), F \in S_n^0(\Delta)$, meet exactly in mnT distinct points, and the piecewise algebraic curves G and F meet exactly in rnT points of these mnT points, where $G \in S_r^0(\Delta)$.*

(1) *If Δ has no interior vertices or Δ is a star partition, then there must exist $G_{m-r} \in S_{m-r}^0(\Delta)$, such that the piecewise algebraic curve G_{m-r} passes through the $(m-r)nT$ remainder points.*

(2) *If Δ is a cross-cut partition, then there must exist $G_{m-r} = A_{m-r} + Q_{m-r}^{\Delta}$, where $G_{m-r}, Q_{m-r}^{\Delta} \in \mathbf{P}_{m-r}(\Delta)$, and $A_{m-r} \in S_{m-r}^0(\Delta)$, such that the curve $G_{m-r} = 0$ passes through the $(m-r)nT$ remainder points.*

Proof. (1) If Δ has no interior vertices or Δ is a star partition, by Lemma 2.3 and Lemma 2.4, we have

$$H = G_{m-r}G + F_{m-n}F,$$

where $G_{m-r} \in S^0_{m-r}(\Delta)$ and $F_{m-n} \in S^0_{m-n}(\Delta)$. Since the piecewise algebraic curve G contains rnT points of those mnT points, the piecewise algebraic curve G_{m-r} contains the $(m-r)nT$ remainder points.

(2) If Δ is a cross-cut partition, by Lemma 2.5, we have

$$H = G_{m-r}G + F_{m-n}F,$$

$$G_{m-r} = A_{m-r} + Q^{\Delta}_{m-r},$$

where $G_{m-r}, Q^{\Delta}_{m-r} \in \mathbf{P}_{m-r}(\Delta)$, $A_{m-r} \in S^0_{m-r}(\Delta)$. Since the piecewise algebraic curve G contains rnT points of those mnT points, the curve $G_{m-r}=0$ contains the $(m-r)nT$ remainder points. □

Theorem 3.3. *Let m and n be natural numbers. Suppose that Δ_c has no interior vertices. If two piecewise algebraic curves G and F, $G \in S^0_m(\Delta), F \in S^0_n(\Delta)$, meet exactly in a collection of mnT distinct points Γ, then for any integer $k \geqslant 0$ the number $h_\Gamma(k)$ of conditions imposed by Γ on $S^0_k(\Delta)$ is independent of the choice of the piecewise algebraic curves G and F, and it can be written explicitly as*

$$h_\Gamma(k,0) = C(k) + (T-1)C(k-1),$$

where $C(k) = C^2_{k+2} - C^2_{k-m+2} - C^2_{k-n+2} + C^2_{k-m-n+2}$ and $C^2_a = 0$ if $a < 2$.

Proof. It is clear that the dimension of $S^0_k(\Delta)$ is $C^2_{k+2} + (T-1)C^2_{k+1}$. By using Lemma 2.3, the formula for $h_\Gamma(k,0)$ given above is easy: it just says that the vector space of forms of $S^0_k(\Delta)$ which can be written as $AF+BG$ is the sum of the dimension of the space of possible A's and the space of possible B's, minus the dimension of the space of pairs (A,B) such that $AF+BG=0$. Since the piecewise algebraic curves G and F meet exactly in mnT distinct points, F and G have no local common factors. Then $AF+BG=0$ can hold only if $A=QG, B=-QF$, where $Q \in \mathbf{P}_{k-m-n}(\Delta)$. Let δ_i, δ_{i+1} be two adjacent cells in the flow $\vec{C}$ of Δ [11], and $l_i=0$ be the edge between δ_i and δ_{i+1}. Thus we have

$$A_i = Q_iG_i, A_{i+1} = Q_{i+1}G_{i+1}, \tag{3}$$

$$B_i = -Q_iF_i, B_{i+1} = -Q_{i+1}G_{i+1}, \tag{4}$$

$$F_{i+1} = F_i + q_il_i, \tag{5}$$

$$G_{i+1} = G_i + p_il_i, \tag{6}$$

where q_i and p_i are the smoothing cofactors of the edge l_j of F and G respectively. From (3)–(6) and by direct computation, we get $Q_{i+1} = Q_i + \omega_il_i$, where $\omega_i \in \mathbf{P}_{k-m-n-1}$. Then we have $Q \in S^0_{k-m-n}(\Delta)$. Thus the dimension of the space of these pairs is the dimension of the space of $S^0_{k-m-n}(\Delta)$. By Definition 2.2, the number $h_\Gamma(k,0)$ of conditions imposed by Γ on $S^0_k(\Delta)$ can be written as

$$h_\Gamma(k,0) = \dim(S^0_k(\Delta)) - \dim(S^0_{k-m}(\Delta)) - \dim(S^0_{k-n}(\Delta)) + \dim(S^0_{k-m-n}(\Delta)),$$

i.e.,

$$h_\Gamma(k,0) = C(k) + (T-1)C(k-1),$$

where $C(k) = C^2_{k+2} - C^2_{k-m+2} - C^2_{k-n+2} + C^2_{k-m-n+2}$ and $C^2_a = 0$ if $a < 2$. □

Theorem 3.4. *Suppose that $BN(m,0;n,0;\Delta) = n^2T$, and two piecewise algebraic curves G and F, $G \in S^0_m(\Delta), F \in S^0_n(\Delta)$, meet exactly in a collection of n^2T distinct points Γ. If $\Gamma' \subset \Gamma$ and $h_{\Gamma'}(n,0) = h_\Gamma(n,0)$, then the piecewise algebraic curve H, $H \in S^0_n(\Delta)$, containing all of Γ' must contain all of Γ.*

Proof. By Definition 2.2, if $h_{\Gamma'}(n,0) = h_\Gamma(n,0)$, there exists a subset of Γ', denoted by $\tilde{\Gamma}$, consisting of $h_\Gamma(n,0)$ points of Γ', such that $\tilde{\Gamma}$ imposes independent conditions on $S^0_n(\Delta)$. Thus $\Gamma/\tilde{\Gamma}$ imposes the conditions on $S^0_n(\Delta)$ dependent on $\tilde{\Gamma}$. Therefore, if the piecewise algebraic curve H, $H \in S^0_n(\Delta)$, passes through all of Γ', then it passes through all of Γ as well. □

Denote by Δ_l the partition consisting of only a line which divides the domain Ω into two parts δ_1, and δ_2. By using Theorem 3.4, we obtain an interesting result as follows, which is similar to the famous Chasles's Theorem of classical algebraic geometry [2,8].

Theorem 3.5. *Suppose two piecewise algebraic curves G and F, $G \in S^0_2(\Delta_l), F \in S^0_2(\Delta_l)$, meet exactly in 8 distinct points $p_1, \dots, p_8$. If the piecewise algebraic curve H, $H \in S^0_2(\Delta_l)$, passes through $p_1, \dots, p_7$, then it passes through the points p_8 as well.*

In order to prove Theorem 3.5, we should prove the following lemma first.

Lemma 3.1. *Let $\Gamma' = \{p_1, \dots, p_7\} \subset \Omega$ be any collection of 7 distinct points, and each cell of Δ_l contain 5 points of Γ' at most. The points of Γ' fail to impose independent conditions on $S^0_2(\Delta_l)$, i.e., $h_{\Gamma'}(2,0) \neq 7$ if and only if Γ' is contained in a nonzero piecewise linear algebraic curve.*

Proof. The "if" direction of the Lemma is easy: If Γ' is contained in a piecewise linear algebraic curve f, then by Bezout's theorem of piecewise algebraic curves any piecewise algebraic curve g, $g \in S^0_2(\Delta_l)$, containing Γ' must have local common components with the piecewise linear algebraic curve f. Thus the codimension of the subspace of $S^0_2(\Delta_l)$ having local common factors with f is $9 - 3 = 6$ at most. Then the points of Γ' fail to impose independent conditions on $S^0_2(\Delta_l)$.

For the "only if" direction, the statement that Γ' itself fails to impose independent conditions on $S^0_2(\Delta_l)$ amounts to saying that *any piecewise algebraic curve $g = 0$, $g \in S^0_2(\Delta_l)$, containing all but one of the points of Γ' contains Γ'*. Using the properties of the piecewise algebraic curve, if 3 points of Γ' do not lie on the same cell of Δ_l, then they must be contained in a nonzero piecewise linear algebraic curve. Suppose that any nonzero piecewise linear algebraic curve contains 3 points of Γ' at most. Using the fact that each cell of Δ_l contains 5 points of Γ' at most, there must exist a collection of 6 points of Γ', denoted by $\tilde{\Gamma}$, such that there are 3 points of $\tilde{\Gamma}$ contained in a nonzero piecewise linear algebraic curve f while the other points of $\tilde{\Gamma}$ contained in another nonzero piecewise linear algebraic curve g. Let $h = fg$. We can see that $h \in S^0_2(\Delta_l)$ and the piecewise algebraic curve h passes through all but exactly one point of Γ'. Since the curve $h = 0$ must contain all of Γ', it is impossible.

Thus at least 4 points of Γ' are contained in a nonzero piecewise linear algebraic curve. The similar arguments, using the fact that each cell of Δ_l contains 5 points of Γ' at most, work in the cases of 4, 5 and 6 points of Γ' contained in a nonzero piecewise linear algebraic curve, respectively. Then we get the conclusion. □

Proof of Theorem 3.5. Let $\Gamma = \{p_1,\dots,p_8\}$, and $\Gamma' = \{p_1,\dots,p_7\}$. Obviously, each cell of Δ_l contains no more than 5 points of Γ'. Suppose that Γ' is contained in a nonzero piecewise linear algebraic curve. From $BN(2,0;1,0;\Delta_l)=4$, then the piecewise algebraic curves G and F have local common components. Since the piecewise algebraic curves G and F meet exactly 8 distinct points, it is impossible. From Lemma 3.1, we have $h_{\Gamma'}(2,0)=7=h_{\Gamma}(2,0)$. By Theorem 3.4, if piecewise algebraic curve H, $H\in S_2^0(\Delta_l)$, passes through Γ', then it passes through p_8 as well. □

Corollary 3.1. *Suppose piecewise algebraic curves G and F, $G\in S_2^0(\Delta_l), F\in S_2^0(\Delta_l)$, meet exactly in 8 distinct points $p_1,\dots,p_8$, and each cell of Δ_l do not contain all of the points $p_1,\dots,p_4$. If the points $p_1,\dots,p_4$ are contained in a nonzero piecewise linear algebraic curve, then the points $p_5,\dots,p_8$ are contained in another nonzero piecewise linear algebraic curve.*

Proof. Let the points $p_1,\dots,p_4$ be contained in a nonzero piecewise linear algebraic curve f. According to the assumption conditions, there must exist 3 points of the points $p_5,\dots,p_8$ lie on another nonzero piecewise linear algebraic curve. Without generality, we assume that the points p_5, p_6, p_7 lie on the nonzero piecewise linear algebraic curve g. Let $h=fg$. Then we have $h\in S_2^0(\Delta_l)$ and the piecewise algebraic curve h passes $p_1,\dots,p_7$. By Theorem 3.5, the piecewise algebraic curve h passes p_8 as well. Suppose the piecewise linear algebraic curve f contains the point p_8. From $BN(2,0;1,0;\Delta_l)=4$, then the curves $G=0$ and $F=0$ have local common components. Since $G=0$ and $F=0$ meet exactly 8 distinct points, it is impossible. Thus the piecewise linear algebraic curve g passes through p_8. □

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